Remote Lesson 7.4

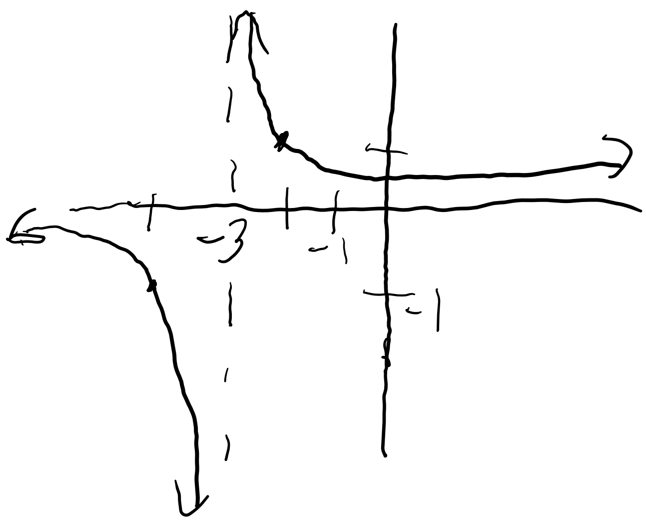
Rational functions are functions that can be stated in the form $\frac{p(x)}{q(x)}$, $where\ q(x) \neq 0$.

We will now be taking our parent reciprocal function, $f(x) = \frac{1}{x}$, and branching out to a ratio of polynomials.

Let's look at some things we already know

Ex) $f(x) = \frac{1}{x+3}$.

In terms of analysis, we recognize that x = 3 must be removed from the domain. What we know about graph shifting, we know that the parent function has been moved 3 units to the left. So there is essential discontinuity at x = -3. Our sketch would look something like this.



Notice we still have a horizontal asymptote at y=0,

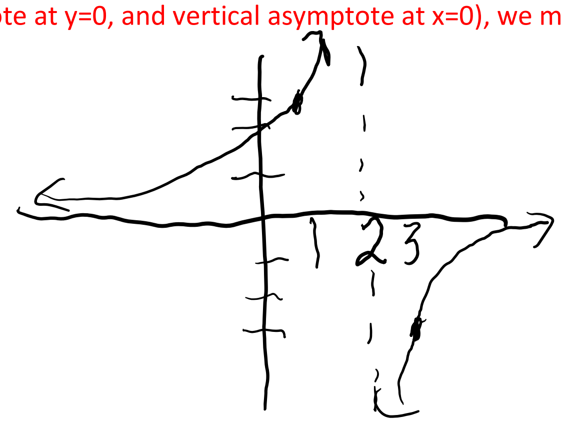
so our end behavior would be unchanged from the parent.

You should be able to analyze this function.

Ex) Describe how the graph of the given function can be transformed from the reciprocal function. Identify the horizontal and vertical asymptotes. Give the end behavior of the function.x

a. $f(x) = \frac{-3}{x-2}$. We should recognize the transformations as a horizontal shift right 2, a vertical stretch of 3, and reflection over the x-axis. Using the anchors of the parent (the points (1,1), (-1,-1), horizontal asymptote at y=0, and vertical asymptote at x=0), we make the transformation.

V Asymptote: x = 2
H Asymptote: y= 0
$\lim_{x\to\infty} f(x) = 0$
$\lim_{x\to-\infty} f(x) = 0$



b. $f(x) = \frac{4x+2}{x-3}$ So what we have to recognize here is we cannot describe transformations from $\frac{1}{x}$ because of the "x" term in the numerator. We will have to divide this function before we can discuss transformations.

3	4	2
		12
	4	14

We divided a degree 1 polynomial by a degree 1 polynomial, so our answer is

$4 + \frac{14}{x-3}$ *or looking more like the parent,* $\frac{14}{x-3} + 4.$

With the x removed from the numerator we can now discuss transformations.

H shift 3 right, V stretch 14, and V shift up 4.

Think about the vertical shift moving all anchors up 4, including the horizontal asymptote.

V Asymptote: x=3

H Asymptote: y=4

$\lim_{x\to\infty} f(x) = 4$

$\lim_{x\to-\infty} f(x) = 4$

**Notice the relationship between the horizontal asymptote and end behavior!

Also notice, that if we look at the degree of the numerator as compared to the degree of the degree of the denominator.

Observations like this lead us to thinking about ways to determine asymptotes and end behavior of rational functions without graphing them. The following gives us guidelines into just such ideas.

GRAPHS OF RATIONAL FUNCTIONS

Given a rational function $f(x) = \frac{p(x)}{q(x)}$, where p(x) and q(x) have no common factors,

1. Vertical Asymptotes are the zeros of q(x)
2. End Behavior Asymptotes
   a. If the degree of p is less than the degree of q, then the end behavior asymptote is y=0
   b. If the degree of p is equal to the degree of q, the EBA can be found by dividing the leading terms of p and q.
   c. If the degree of p is greater than the degree of q, the EBA is found by dividing p(x) by q(x).
3. X-intercepts occur at the zeros of the numerator.
4. Y-intercepts are the value of f(0).

Notice we have substituted the term horizontal asymptote with end behavior asymptote, because we will discover, sometimes horizontal asymptotes are not truly asymptotes, and because not all asymptotes are horizontal.

Ex) Identify the intercepts and asymptotes of each rational function

1. $f(x) = \frac{3x-7}{x^2-2x-8}$

   Write in factored form: $f(x) = \frac{3x-7}{(x-4)(x+2)}$

   Using the above info: VA: x=4, x = -2

   EBA: y = 0 (degree of numerator < degree denominator)

   x-int: $3x - 7 = 0$, $\boldsymbol{x = \frac{7}{3}}$

   y-int: $\frac{3(0)-7}{0^2-2(0)-8}$, $\boldsymbol{y = \frac{7}{8}}$ (use the original function here-easier)

2. $f(x) = \frac{4x^2-1}{x^2-16}$

   $f(x) = \frac{(2x-1)(2x+1)}{(x+4)(x-4)}$
   VA: x=4, x=-4
   EBA: y = 4 (degree numerator = degree denominator, so divide lead terms)
   x-int: $x = \pm\frac{1}{2}$
   y-int: y= $\frac{1}{16}$

3. $f(x) = \frac{3x^3-2x^2+x-1}{x-5}$

Parts of this can be done with a calculator
VA: x = 5 (no calculator needed here)
EBA: Because the degree of the numerator > degree denominator, we will divide

$$\begin{array}{c|cccc} 5 & 3 & -2 & 1 & -1 \\ & & 15 & 65 & 330 \\ \hline & 3 & 13 & 66 & 329 \end{array}$$

$\boldsymbol{EBA: y = 3x^2 + 13x + 66}$ Notice only the quotient is the EBA. The remainder is what provides space between the function and the asymptote. Also notice the EBA is not horizontal. Graph the original function in your calculator and ZOOM fit to see this. If the numerator is not linear, you will have to do long division to find this EBA. Please find a video and review long division.

x-int: $x \approx .784$ (a calculator can be used here)
y-int: $y = \frac{1}{5}$ (no calculator is needed here)

The process discussed in the examples above will be the beginning of the next lesson. Please make sure you have studied this material well

HOMEWORK

In questions 1-4, describe how the graph of the given function can be obtained by transforming the graph of $g(x) = \frac{1}{x}$. Identify the horizontal and vertical asymptotes of the function. Give the end behavior.

1. $f(x) = \frac{1}{x-3}$
2. $f(x) = -\frac{2}{x+5}$
3. $f(x) = \frac{2x-1}{x+3}$
4. $f(x) = \frac{3x-2}{x-1}$

In questions 5-10, identify the intercepts and asymptotes of the function. Give end behavior

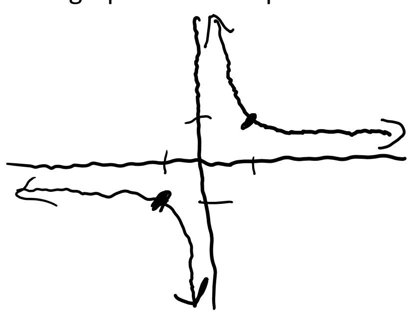
5. $f(x) = \frac{x-2}{x^2-2x-3}$
6. $f(x) = \frac{2}{x^3-x}$
7. $f(x) = \frac{2x^2+x-2}{x^2-1}$

8. $f(x) = \frac{x^2-2x+3}{x+2}$
9. $f(x) = \frac{-3x^2+x+12}{x^2-4}$
10. $\frac{x^2-3x-7}{x+3}$

LESSON ENDS HERE (OPTIONAL MATERIAL BELOW)

OPTIONAL NOTES ON LIMITS

Look at the graph of the reciprocal function



We have discussed end behavior limits. We know that in end behavior, x MUST approach infinity and negative infinity; however, if we leave end behavior, there are other limits we can discuss. What a function does near a point or area of discontinuity is also of interest.

The graph of $f(x) = \frac{1}{x}$, there is a VA (essential discontinuity) at x = 0.

We can look at $\lim_{x \to 0^+} f(x)\ and\ \lim_{x \to 0^-} f(x)$ These are read, " the limit of f(x) as x approaches 0 from the right" and " the limit of f(x) as x approaches 0 from the left."

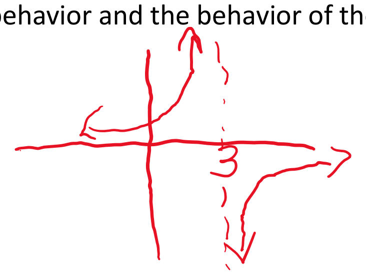
From the right: start at the rightmost side of the graph. Move in closer to the asymptote, staying on the right side of the asymptote. Describe what is happening to the y. So, $\lim_{x \to 0^+} f(x) = \infty$

From the left: start at the leftmost side of the graph. Move in closer to the asymptote, staying on the left side of the asymptote. Describe what is happening to the y. So, $\lim_{x \to 0^-} f(x) = -\infty$

Ex) Describe the end behavior and the behavior of the function near any vertical asymptote if

$f(x) = \frac{-1}{x-3}$

Graph the function:



End Behavior: $\lim_{x \to \infty} f(x) = 0$; $\lim_{x \to -\infty} f(x) = 0$. (deg numerator < degree denominator so EBA y=0)

Vertical Asymptote: $\lim_{x \to 3^+} f(x) = -\infty$; $\lim_{x \to 3^-} f(x) = \infty$

If these make sense, you can also do p. 246 11-22