Remote Lesson 7.2
ZEROS OF POLYNOMIAL FUNCTIONS

Synthetic Division: Used when a polynomial P(x) is divided by (x-r). *Notice, synthetic division can and should be used when the denominator is linear (1st degree).*

Ex1: Find the remainder

$$\frac{x^3 - 7x - 4}{x + 1}$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -4 \\ & \downarrow & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 2 \end{array}$$

Please find a video to review synthetic division if necessary. Quick reminders. The 0 in the top row represents the coefficient of x^2, because the polynomial must be written in descending order. Also, remember what is happening- we are dividing a 3rd degree polynomial by a 1st degree polynomial, so our quotient (answer) will be a 2nd degree polynomial and its coefficients lie on the bottom line.

But if all we need is the remainder, we have a better way.

Remainder Theorem: If a polynomial P(x) is divided by (x-r), the remainder is P(r).

****Return to example 1 and find the remainder using the theorem**
So here, $P(x) = x^3 - 7x - 4$, *and* $(x - r)$ *would be* $(x + 1)$, *making* $r = -1$

So, $P(r) = P(-1) = (-1)^3 - 7(-1) - 4 = -1 + 7 - 4 = \mathbf{2}$

We get the same answer, so the question you want to ask yourself is when would we only want to know a remainder?

Factor Theorem: (x-r) is a factor of P(x) *iff* P(r) = 0.

Ex2: Is (x-5) a factor of $x^3 - 4x^2 - 7x + 10$?
$P(x) = x^3 - 4x^2 - 7x + 10$ *and* $(x - r)$ *is* $(x - 5)$, *so* $r = 5$

$P(5) = 5^3 - 4(5)^2 - 7(5) + 10$, *so* $P(5) = 125 - 100 - 35 + 10 = 0$

Final Answer: $\boldsymbol{(x - 5)}$ ***is a factor of*** $\boldsymbol{x^3 - 4x^2 - 7x + 10}$

So to answer the earlier question, we would want to know only a remainder to see if we have a factor. IF we have a factor, it would make sense to do the division at the beginning. If not, then we would save ourselves from having to divide.

<u>We need to consider the implications of the following statements:</u>

1. **x = k is a solution (root) of f(x) = 0.** Solutions or roots imply any answers within the complex number system (includes imaginaries). Answers in the form x=
2. **k is a zero of f.** *Zeros TYPICALLY refer to x-intercepts (though some sources now lump them in with roots and solutions). If they are considered x-intercepts, that would limit us to the answers only in the real number system.*
3. **k is an x-intercept of the graph y = f(x).** *This would be the equivalent of #2*
4. **(x-k) is a factor of f(x).** *As we just saw in the last example problem, factors are in the form (x-r). So if (x-5) is a factor of f(x), then x=5 is a root of f(x).*

So as we look at polynomial equations of a degree higher than 2, we have to look at the idea that factors come from division. Below is one method we will consider.

> **<u>Rational Root Theorem</u>: Given a polynomial P(x) written in descending order, if $\frac{p}{q}$ is a root of the equation, then p is a factor of the constant and q is a factor of the leading coefficient.**

Ex3 Find ALL roots of $2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3 = 0$

*So, keep in mind, the above theorem will only help us find **rational** solutions. No irrationals (like $\sqrt{5}$) and no imaginaries. This theorem gives us a place to start. Think about the quadratic formula—can we find irrational answers there (YES!). Can we find imaginaries there (YES!). So, our goal will be to divide this down to a quadratic (hopefully) and from there use factoring or the quadratic formula to find remaining solutions in case they are irrational or imaginary.*

Let's get started. The polynomial must be written in descending order. Our p values will come from the constant, 3, and our q values will come from the leading coefficient, 2. Our list of *possible* rational roots will be

$$\frac{p}{q} = \frac{factors\ of\ 3}{factors\ of\ 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}.$$

The theorem tells us these are the only numbers that could possibly divide evenly into P(x). We will begin with the easiest possible root, 1

1	2	3	-6	6	-8	3
		2	5	-1	5	-3
	2	5	-1	5	-3	0

We have our first root, x=1. We also now know that both $(x-1)$ *and* $(2x^4+5x^3-x^2+5x-3)$ are factors of P(x). Remember, we are trying to get our polynomial down to quadratic. Let's think transitively for a moment. Since, $(2x^4+5x^3-x^2+5x-3)$ is a factor of P(x), wouldn't it stand to reason that anything that is a factor of $(2x^4+5x^3-x^2+5x-3)$, would also be a factor of P(X)? WE DO NOT NEED TO CHANGE OUR $\frac{p}{q}$ LIST! However we will now try to find factors of $(2x^4+5x^3-x^2+5x-3)$. So back to it

Let's try -3 from our list.

$$\begin{array}{r|rrrrr}
-3 & 2 & 5 & -1 & 5 & -3 \\
 & & -6 & 3 & -6 & 3 \\
\hline
 & 2 & -1 & 2 & -1 & \boxed{0}
\end{array}$$

We have our second root of x=-3 and we also now know that $2x^3-x^2-2x-1$ is a factor of P(x). *Note, had we not gotten zero as our remainder, we would have kept trying numbers from the list until we did.*

Following our transitivity thought we will now look for factors of $2x^3-x^2-2x-1$. Looking at 2s and 1s next to each other, $\frac{1}{2}$ seems like a logical choice from our $\frac{p}{q}$ list.

$$\begin{array}{r|rrrr}
\frac{1}{2} & 2 & -1 & 2 & -1 \\
 & & 1 & 0 & 1 \\
\hline
 & 2 & 0 & 2 & \boxed{0}
\end{array}$$

We have found our 3rd root! Not only that, notice our remaining polynomial is a quadratic (3 terms). We MUST at this point pull out the quadratic and use known methods (factoring, quadratic formula, etc) to solve in irrationals or imaginaries exist as roots of this polynomial. We will now solve $2x^2+0x+2=0$ using the method of your choosing

$2x^2+2=0$
$2x^2=-2$
$x^2=-1$
$x=\pm i$

We have solved our 5th degree polynomial. We have found 5 roots
Final answer: $x=-3,\frac{1}{2},1,\pm i$

Let's try one more

Ex4 Find ALL roots of $2x^3 + 2x^2 + 8x + 8 = 0$

$$\frac{p}{q} = \frac{factors\ of\ 8}{factors\ of\ 2} = \frac{\pm1, \pm2, \pm4, \pm8}{\pm1, \pm2} = \pm1, \pm2, \pm4, \pm8, \pm\frac{1}{2}$$

Try 1

$$\begin{array}{r|rrrr} 1 & 2 & 2 & 8 & 8 \\ & & 2 & 4 & 12 \\ \hline & 2 & 4 & 12 & 20 \end{array}$$

Doesn't work…we will try another

Try -1

$$\begin{array}{r|rrrr} -1 & 2 & 2 & 8 & 8 \\ & & -2 & 0 & -8 \\ \hline & 2 & 0 & 8 & 0 \end{array}$$

We have our first root at x=-1.
We are also down to quadratic
We will pull it out and solve

$2x^2 + 8 = 0$
$2x^2 = -8$
$x^2 = -4$
$x = \pm 2i$
Final Answer: $x = -1, \pm 2i$

So both of these examples support The Fundamental Theorem of Algebra which tells us that within the complex number system, a polynomial of degree n will have n solutions.

HOWEVER, the process almost inspires more questions than it answers. The first question might be how do we know which numbers will work from the list or are we just guessing? Quickly followed by what are we going to do with a long list of factors (for instance a constant of 36 and a leading coefficient of 72).

This is going to be, to some extent, a trial and error process like factoring
Below are some ideas to help rein in the process a little.

<u>Upper and Lower Bounds</u>: Let f be a polynomial function of degree n $\geq$ 1 with a positive leading coefficient. Suppose f(x) is divided by (x – k)

- **If k $\geq$ 0 and every number in the last line is nonnegative, then k is an <u>upper bound</u> for the real zeros of f.**
- **If k $\leq$ 0 and the numbers in the last line are alternately nonnegative and nonpositive, then k is a <u>lower bound</u> for the zeros of f**

*****Look at the last example. How would this help in our process? This was our question**

Find ALL roots of $2x^3 + 2x^2 + 8x + 8 = 0$

This was our list of possible roots $\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}$

We tried 1

$$\begin{array}{r|rrrr} 1 & 2 & 2 & 8 & 8 \\ & & 2 & 4 & 12 \\ \hline & 2 & 4 & 12 & 20 \end{array}$$

Reading about bounds, the k value we have is 1
The "last line" is 2, 4, 12, 20
All of our numbers are nonnegative
K=1 is an upper bound for the zeros of f.

The significance of an upper bound is that none of the numbers in our $\frac{p}{q}$ list that are larger than 1 are going to give a zero remainder. So we will not need to try 2, 4, and 8 from our list, they will not work.

Let's look at the lower bound end of things. These would be our negative $\frac{p}{q}$ values

We tried -1

$$\begin{array}{r|rrrr} -1 & 2 & 2 & 8 & 8 \\ & & -2 & 0 & -8 \\ \hline & 2 & 0 & 8 & 0 \end{array}$$

Our k value is -1
The "last line" is 2, 0, 8, 0
These numbers are alternating nonnegative and Nonpositive. (zero fits in either category so we think 2 is nonnegative and 0 is nonpositive, etc
-1 is a lower bound for the zeros of f

So no numbers from the list that are less than -1 will not yield factors.

This would be part of the "narrowing" process for longer $\frac{p}{q}$ lists. As a means of working through the problem, always start with the easiest values—try 1, -1, even 2, -2. If the list is long, move to the middle and do the division. See if you have a bound. If you do, you have eliminated a set of numbers from the long list. This is just to help in the trial and error process.

In the book, the questions look like this.

Ex) use synthetic division to prove that the number, *k*, is an upper bound for the zeros of f.
$\boldsymbol{k = 5; f(x) = 2x^3 - 5x^2 - 5x - 1}$

So they instruct you the number to try. The point is to make sure you can recognize it is a bound.

$$\begin{array}{r|rrrr} 5 & 2 & -5 & -5 & -1 \\ & & 10 & 25 & 100 \\ \hline & 2 & 5 & 20 & 99 \end{array}$$

In the bottom line, all numbers are nonnegative (no sign change) so 5 is an upper bound.

This could be helpful for when we are using our calculator and wanting to make sure we are capturing all the real zeros on our screen. We can go to the window and test our x-max for an upper bound and our x-min for a lower bound. If we can establish both bounds, then we know we have all of the real zeros of the function

In a previous math course, you examined the discriminant to determine what type of solutions you were going to get from a quadratic. As you recall the discriminant is the value of $\boldsymbol{b^2 - 4ac}$

For example you may have had a quadratic like, $\boldsymbol{f(x) = 2x^2 - 7x + 9}$. In that quadratic, a=2, b=-7 and c=9. So the discriminant would have a value of $\boldsymbol{(-7)^2 - 4(2)(9) = -23}$. -23 is the value under the square root, so you now know that you are going to get 2 complex solutions when you go to solve. Finding the discriminant is just a quick way to "diagnose" the type of solution.

Below, is the method used for polynomial functions whose degree is higher than 2.

OPTIONAL!

Descartes' Rule of Signs
Given a polynomial equation p(x) = 0, the number of positive real roots is the same as the number of sign changes in p(x) or is less than this by an even number. The number of negative real roots is the same as the number of sign changes in p(-x) or is less than this by an even number.

So as you read this rule of signs, keep in mind that is finds the possible *real* roots. This does NOT account for complex solutions containing imaginary numbers. Complex solutions always come in pairs (if a+bi is a root, then a-bi is also a root). Because there are 2, we have to say "or less than this by an even number." It leaves room for possible imaginary solutions.

Ex) Find the number of positive and negative real roots.

1. **P(x) =** $x^4 - 3x^3 - 2x^2 + 3x + 8$
   There are two sign changes in P(x), so P(x) has 2 or 0 positive real roots

   $\boldsymbol{P(-x) = (-x)^4 - 3(-x)^3 - 2(-x)^2 + 3x + 8}$
   $\boldsymbol{P(-x) = x^4 + 3x^3 - 2x^2 + 3x + 8}$

   P(-x) has 2 sign changes, so P(-x) has 2 or 0 negative real roots

We have answered the question. We have now different scenarios (you are not required to name these, but it helps with the understanding of the Rule). Our degree four polynomial has either

a) 2 positive real roots and 2 negative real roots OR
b) 2 positive real roots, 0 negative real roots, and 2 complex roots OR
c) 0 positive real roots, 2 negative real roots, and 2 complex roots OR
d) 0 positive real roots, 0 negative real roots and 4 complex roots

This can also help in having long lists of possible rational solutions. It enables us to "troubleshoot" the polynomial.

2. **P(x) =** $\boldsymbol{x^3 - 4x^2 + x - 2}$

   P(x) has 3 sign changes so P(x) has 3 or 1 positive real roots

   $P(-x) = -x^3 - 4x^2 - x - 2$

   P(-x) has no sign changes so P(x) has no negative real roots

HW pp 216-218 1-55 every other odd