Remote Lesson 6.5

Composition and Inverses

Review: You should remember how to do these

Example: Given $f(x) = 2x^2 - 1$ *and* $g(x) = 4x + 7$, find

1. $f(4)$=31 2. g(-2)=-1 3. f(g(5))=1457 4. g(f(5))=203

5. f(g(x)) 6. g(f(x)) *Note f(g(x)) and g(f(x)) are called* ***compositions***

$$5.\ f(g(x)) = f(4x+7) = 2(4x+7)^2 - 1 = \mathbf{32x^2 + 112x + 97}$$

$$6.\ g(f(x)) = g(2x^2-1) = 4(2x^2-1) + 7 = \mathbf{8x^2 + 3}$$

Now you try: Given $f(x) = x^2 + 4$ *and* $g(x) = \sqrt{x}$, find

1. $f(g(x))$=$x + 4$
2. $g(f(x))$=$\sqrt{x^2 + 4}$

Now let's consider notations. The following notations ask the same question

1. $(\boldsymbol{f} + \boldsymbol{g})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})$
2. $(\boldsymbol{f} - \boldsymbol{g})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})$
3. $(\boldsymbol{fg})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})$
4. $\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)\boldsymbol{x} = \frac{\boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{g}(\boldsymbol{x})}, \boldsymbol{provided}\ \boldsymbol{g}(\boldsymbol{x}) \neq \mathbf{0}$
5. $(\boldsymbol{f} \circ \boldsymbol{g})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$

Example: Given $f(x) = (x-1)^2$ *and* $g(x) = 3 - x, find\ f + g, f - g, fg, and\ \frac{f}{g}$. Then state the domain.

1. $f + g = (x-1)^2 + 3 - x = x^2 - 3x + 4, domain\ (-\infty, \infty)$
2. $f - g = (x-1)^2 - (3 - x) = x^2 - x - 2, domain(-\infty, \infty)$
3. $fg = (x-1)^2(3 - x) = -x^3 + 5x^2 - 7x + 3, domain(-\infty, \infty)$
4. $\frac{f}{g} = \frac{(x-1)^2}{3-x}, domain\ (-\infty, 3) \cup (3, \infty)$

Now let's look at the domain of a composition.

Example $given\ f(x) = x^2 - 1\ and\ g(x) = \sqrt{x}, find\ f \circ g\ and\ state\ the\ domain$

So we will begin by finding f(g(x))

$$f(\sqrt{x}) = (\sqrt{x})^2 - 1$$

$$f(g(x)) = x - 1$$

To find the domain of a composition, we will follow a 3 step process (our solution in red)

1. **Find the domain of the innermost function:** $\boldsymbol{D_g: x \geq 0\ (use\ inequalities\ in\ steps\ 1,2)}$
2. **Find the domain of final answer to composition** $\boldsymbol{D_{ans}: all\ reals}$
3. **Graph both on a number line and the intersection is the domain:** $\boldsymbol{D_{f\circ g}: [0, \infty)}$

Final answer in intervals

Now lets go back to our example, $find\ g \circ f\ and\ state\ the\ domain$

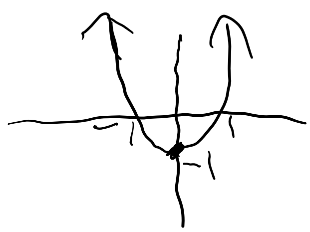
We will find g(f(x))

$$g(x^2 - 1) = \sqrt{x^2 - 1}$$

$$g \circ f = \sqrt{x^2 - 1}$$

Now domain:

1. $D_f: all\ reals$
2. $D_{ans}: x \leq -1\ and\ x \geq 1$ remember here no negative under the radical. $x^2 - 1$ is under the The square root, so consider just the parabola shifted down 1



.

So we are looking at where the graph is positive. $x \leq -1\ and\ x \geq 1$

3. $\boldsymbol{D_{g\circ f}: (-\infty, -1] \cup [1, \infty)}$

YOU CAN SEE THAT YOU NEED TO KEEP PRACTICE DOMAINS IN GENERAL!

Let's look at another one

$$Given\ f(x) = x^2 - 1\ and\ g(x) = \frac{1}{x-1} find\ f \circ g\ and\ state\ domain$$

$$f(g(x)) = (\frac{1}{x-1})^2 - 1$$

$$\boldsymbol{f \circ g = \frac{1}{(x-1)^2} - 1}$$

Domain

1. $D_g: x \neq 1$
2. $D_{ans}: x \neq 1$
3. $D_{f \circ g}: (-\infty, 1) \cup (1, \infty)$

Now you try: using the same functions, find $g \circ f\ and\ state\ domain$

$$g(f(x)) = \frac{1}{x^2 - 1 - 1}$$

$$\boldsymbol{g \circ f = \frac{1}{x^2 - 2}}$$

Domain

1. $D_f: all\ reals$
2. $D_{ans}: x \neq \pm\sqrt{2}$
3. **$D_{g \circ f}: (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$**

Challenge Problem:

$$\boldsymbol{Given\ f(x) = \frac{1}{x^2-4}\ and\ g(x) = x + \frac{1}{x}, find\ f \circ g, g \circ f\ and\ state\ the\ domain}$$

Inverses

In units 2 and 3, we spent time specifically looking at inverses of trig functions. We now want to expand the discussion to all functions.

Things we know

1. Given a function, we can find an inverse by switching x and y and then solving the result for y.
2. Graphs of inverses reflect over the line y=x.

But let's look at a function like $f(x) = x^2$. We know this a function as its graph would pass the vertical line test. If we find its inverse: $f(x) = x^2$

$y = x^2$ *(remember f(x) and y are the same)*

$x = y^2$(switch x and y)

$y = \pm\sqrt{x}$

$f^{-1}(x) = \pm\sqrt{x}$ *(change to inverse notation, read "f-inverse of x")*

So if we just talk in the graphing sense, the inverse of $y = x^2$ is $x = y^2$.

$y = x^2$ is a parabola that opens up—***function***

$x = y^2$ is a parabola that opens to the right –***Not a function***

Definition: Suppose f is a real function (domain is contained in the real number system). If f is increasing throughout it's entire domain or decreasing throughout it's entire domain, then f is a <u>one-to-one function</u>.

Name the functions in the toolkit that are one-to-one: Identity, Odd power, Reciprocal, Square Root, exponential, log, and logistic.

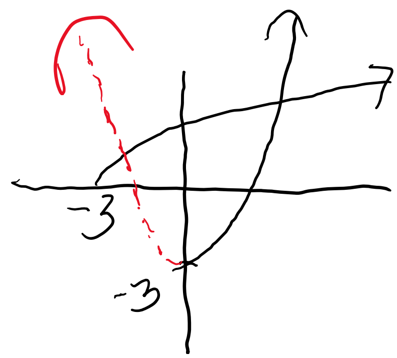
What do we know about the *graphs* of one-to-one functions? They pass both vertical and horizontal line tests!

Conclusion? Only functions that are one-to-one will have inverses that are also functions.

For the rest of these functions, we will have to restrict their domains so that the graphs will be reflections over the line y=x.

Example $\boldsymbol{Given\ f(x) = \sqrt{x+3}, find\ f^{-1}(x).\ State\ any\ restrictions\ inherited\ from\ f.}$

So, the instructions "$\boldsymbol{State\ any\ restrictions\ inherited\ from\ f}$" mean that we need to be aware of the domain and range of the given function.



$y = \sqrt{x+3}$ **, x≥ -3, y≥ 0 (state domain and range)**

$x = \sqrt{y+3}$ **, x≥ 0, y≥ -3 (switch ALL x and y values)**

$x^2 = \sqrt{y+3}^2$ **(square both sides as we solve for y)**

$x^2 = y + 3$

$x^2 - 3 = y$ **(solved for y)**

$f^{-1}(x) = x^2 - 3, x \geq 0, y \geq -3$ **(final answer)**

By restricting the domain, we eliminate the left (red) part of the parabola and our graphs will reflect over the line y=x

****Note, you only need to restrict the domain when you see the directions "$\boldsymbol{State\ any\ restrictions\ inherited\ from\ f}$"**
If you do not see those instructions, simply switch x and y and solve for y.

Example $\boldsymbol{Given\ f(x) = \frac{x+3}{x-2}, find\ f^{-1}(x). State\ any\ restrictions\ inherited\ from\ f.}$

$y = \frac{x+3}{x-2}, x \neq 2, y \neq 1$ **(you can graph on calculator to find range)**

$x = \frac{y+3}{y-2}, x \neq 1, y \neq 2$ **(switch x and y)**

$x(y-2) = y + 3$ **(getting rid of fraction to get y on one side by itself**

$xy - 2x = y + 3$

$xy - y = 3 - 2x$ **(get the y terms on one side)**

$y(x-1) = 3 - 2x$

$y = \frac{3-2x}{x-1}$

$f^{-1}(x) = \frac{-2x+3}{x-1}, x \neq 1, y \neq 2$ **(final answer)**

Confirming Inverses Algebraically: To confirm inverses algebraically, show f(g(x))=g(f(x))=x. In other words find both compositions and show the answer to both is x. They have to equal x because y=x is the reflection line for inverses.

HW: Composition: p 128: 9-19

Inverses: p 129: 43-61 odd